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NETWORK MODELS FOR THE NUMERICAL SOLUTION OF COUPLED ORDINARY NON-LINEAL DIFFERENTIAL EQUATIONS

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Abstract. Many apparently simple problems in mechanics or mechanical engineering, particularly problems related to chaotic systems, are governing by coupled differential equations, generally non-lineal, that have to be solved numerically by specialists in this field. The network model, a tool very used in the last decades for numerical problems in different fields of science and engineering, allows that non-specialists, and even students familiarized with circuits theory, to design networks whose governing equations are just those of the engineering phenomenon, assuming a suitable or formal equivalence between electrical and physical variables. The design of the model, which is composed of a principal network, which implements a balance between the addends of the differential equations, and auxiliary networks to implement the derivative terms, follows a standard procedure. Non-lineal terms of the differential equations are implemented by a controlled source, a kind of device whose operation is quite intuitive. In this communication the models of two characteristic non-lineal mechanical problems are designed step by step with a detailed explanation: the elastic pendulum and the chaotic double pendulum: Solutions are presented graphically by using MATLAB.

1 INTRODUCTION

A large variety of mechanical problems with two or more freedom degrees are formulated by coupled, ordinary differential equations with time as the independent variable. For example: two masses jointed by a spring, a system of masses and pulleys, elastic pendulum, pendulum over a slide bar, double pendulum, etc. Most of these problems are non-lineal or even chaotic, since they contain addends with time harmonic (trigonometric) dependencies or with potential functions of the dependent variables. As a consequence, numerical procedures are required for the solution.

In this paper, we solve this kind of problems by using the network method, a kind of analogy between the real process and one formulated by electric circuits whose equations are formally equivalent to those of the problem. The application of this analogy, per se, can be considered an important goal since it relates phenomena mathematically equivalent. Its use to describe physical processes – even though it has been applied in many areas of science and engineering [1,2], especially in heat conduction [3] – remains under-exploited for problems formulated by coupled ordinary differential equations. The network method has demonstrated to be an efficient tool that provides reliable and computationally fast numerical solutions for a large variety of problems formulated by coupled partial differential equations such as transport through membranes [4], heat transfer [5], inverse problems and fluid flow and solute transport [6], among others. One of the main advantages of using this method is that, if the models are correctly designed, their simulation in suitable software provides (almost) the exact solution of the problem due to the powerful mathematics algorithms implemented in the circuit simulation codes.

The proposed analogy is based on the following [7]. On the one hand, the addends of each differential equation are considered as currents (branches in one of the main circuits) that enter (or leave) the only node of this main circuit, according to their sign; the unknown variable of each differential equation is the voltage at that node. There are as many independent main circuits (even they are couple) as equations define the mathematical model. The first derivative term (dy/dt), one of the branches of the main circuit, is simply the current flowing through a capacitor according to the constitutive equation ic=C(dVc/dt). The successive derivatives are obtained by auxiliary circuits formed by new capacitors, whose capacitance is the coefficients of the term, and a special kind of device contained in the libraries of the software, named controlled source. Once obtained, these derivative terms are transported to the main circuit, where the terms of the equation are balanced in the common node, and again implemented by controlled current-sources according to their sign. The rest of the terms of the equations, such as those depending on the unknown variable and/or its powers (integer or fractional), coupled and independent terms, which must be also balanced at the common node of the main circuit, are implemented by controlled current sources. No restrictions are assumed as regards the order and degree of the equation as well or the kind of non-linearity involved. The model is completed by fixing the initial voltages at the capacitors which are defined by the initial conditions. Once the model is designed no mathematical manipulation is needed; the code Pspice [8] does this work with its powerful computational algorithms.

Two advantages mentioned as regards the network method: (i) no mathematical manipulation (inherent to most of the numerical and analytical methods) is required; the computer code used in this work, Pspice, does the calculations with sophisticated mathematical algorithms (Nagel [9]), and (ii) a few programming rules are necessary to elaborate the network text file, since the number of different electrical devices or components involved is very small.

The sections of the documents are organized as following: a detailed explanation of the network design method followed by its application to two illustrative, selected problems. The section conclusions resume the advantages of the method for this kind of applications.

2 MATHEMATICAL AND NETWORK MODELS

The mathematical model is a system of coupled, ordinary differential equations, with as many equations as dependent variables exist. The equations can be of any order and any grade, with power of real numbers, and may contain coupled terms as well as arbitrary functions of the dependent variables and independent terms.

2.1 Auxiliary circuits

Firstly, we describe the design of the auxiliary networks that implement the derivative terms. Pspice, or any bother code for circuit simulation, contains a group of ideal controlled sources capable of assuming any kind of non-linearity; these, suitably connected with capacitors, provide the auxiliary circuits that implement any derivative term. Four different sources can be used, Figure 1: E is a voltage source whose output is defined (by programming) as an arbitrary function of the voltage at any node (or voltages of any nodes) of the network, while H is a voltage source whose output is proportional to the current of a time-independent voltage source. The other two current-sources, G and F, have similar meanings.

Now, if we call V_j the voltage at node j, the auxiliary network of Figure 2 (a) formed by a capacitor (of capacitance C_a=a_1) and a voltage-controlled voltage-source (whose output voltage is $v_{E1} = V_j$, the same as the input voltage) is able to provide the value $a_1(dV_j/dt)$ since the current through C_a is defined as $i_{Ca} = C_a(dV_j/dt) = a_1(dV_j/dt)$. A new auxiliary loop, formed by the current-controlled voltage-source H_1 and the resistor R_1 (of resistance a_1), provides the first derivative of V_j.

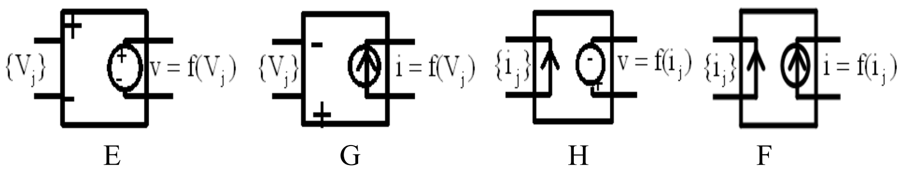

Figure 1: Controlled sources: E: voltage-controlled voltage-source, G: voltage-controlled current-source H: current-controlled voltage-source, and F: current-controlled current-source

The output of H_1 is a voltage whose value is the input current $i_{Vzero,1}$, i.e. the current of the ammeter $V_{zero,1}$ which, in turn, is the current of the capacitor C_a; consequently, the voltage through R_1 is $(1/a_1)a_1(dV_j/dt)=dV_j/dt$, the first derivative function of V_j. Resistor $R_{\infty,1}$ is included to satisfy the continuity criteria required by Pspice. Also, the use of V_{zero} as ammeter is prescribed by the requirements of Pspice: the input current of the controlled sources of type H must be specified as a current coming from a constant voltage source.

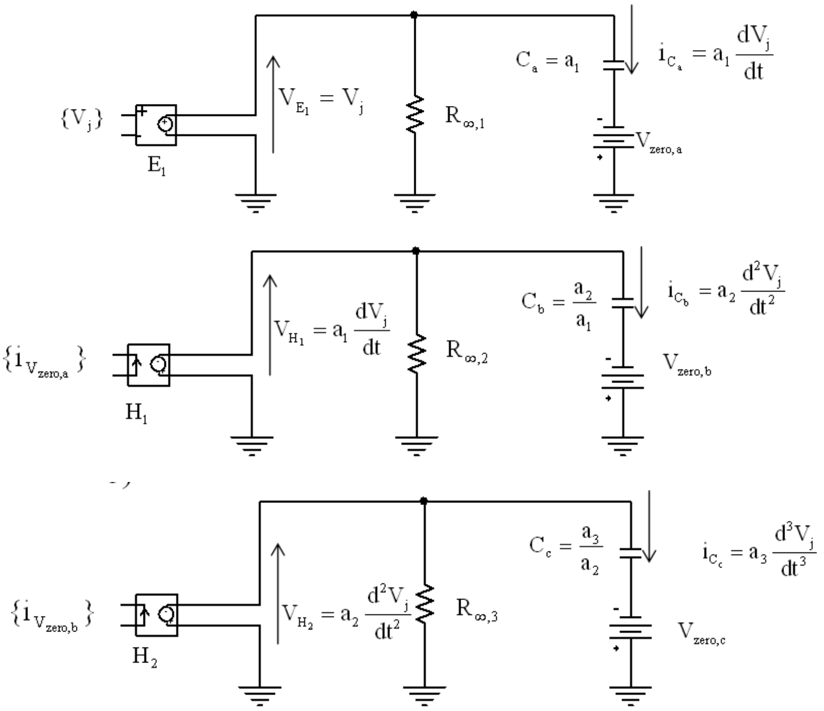

Figure 2: Auxiliary networks to implement the first derivative (a), the second derivative (b) and the third derivative (c)

In the same way, the second derivative of V_j, $a_2(d^2V_j/dt^2)$, is provided by the auxiliary network of Figure 2 (b). The output of the current-controlled voltage-source H_b, $v_{Hb} = i(V_{zero,1}) = a_1(dV_j/dt)$, defines the current through the capacitor C_b (of capacitance a_2/a_1) as $i_{Cb} = C_b(a_1d^2V_j/dt^2) = a_2(d^2V_j/dt^2)$. In addition, H_2 and R_2 (of resistance a_2/a_1) provide the second derivative of V_j (the voltage through the resistor R_2. The following derivative terms are implemented in the same way; Figure 2 (c) shows the network of the third derivative term. Quantities between brackets always denote the control variables that determine the input of the sources.

2.2 Main circuit loops

There are as many main loops as equations or dependent variables in the mathematical model. In turn, each main network is formed by as many branches in parallel as terms of the differential equations. One of the nodes of the main loop is the solution of the related dependent variable while the other is the common reference voltage (earth). Each branch drives a current (whose value is that of the term) that comes in or out of the common node, according to the sign of the term in the equation. The term related to the first derivative term, if it exists, is implemented by a capacitor, while the rest of the derivative terms are implemented by voltage-controlled current-sources; these read their value from their respective auxiliary circuits and introduces them as the output of the controlled source by software. When the derivative term of any order has a degree different from unity (or it is a real number), it is also possible to introduce it by software, as the output of the related controlled source.

The rest of the terms of the differential equation (coupled, independent or other kind of non-lineal terms, such as terms with arbitrary dependencies on the dependent variable) can be implemented in the model by controlled or independent sources. Finally, the independent term is simply implemented by a constant source. A resistor of very high value that does not influence the solution is also located in parallel in the main circuit to satisfy the continuity requirements imposed by the code Pspice.

Whatever be the initial conditions, they are implemented in the model by giving initial voltages at the capacitors. The solutions $y_i(t)$, $1 \leq i \leq n$, are read at the only node of the main circuit (as a consequence of the balance between the currents of the branches, Kirchhoff′s law), while the solutions of successive derivatives can be read at the nodes of the auxiliary circuits.

The network file can be designed by a text editor (text file) or directly by a graphic ambient by means of the option ‘schematics’ contained in the code Spice.

3 APPLICATIONS

3.1 The elastic pendulum

This is the system whose physical scheme is shown in Figure 3. A small body of mass m_o, fixed to a spring of negligible mass, is hanging to the ceiling. The system has degree of freedom, the angle that forms the spring line with the vertical or equilibrium line, θ, and the distance from the mass to the fixing ceiling point, r.

The governing equations are

$$\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - g\left(\cos\theta\right) + \frac{k}{m_o}\left(r - r_o\right) = 0 \tag{1}$$

$$r\left(\frac{d^2\theta}{dt^2}\right) + 2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + g(\text{sen}\theta) = 0 \tag{2}$$

where t is time, k the constant of the spring, g the gravitational acceleration, and r_o the initial location of the mass with respect to the fixing ceiling point. To simplify, we will assume that $r_o = l_o + d_o$, being l_o the length of the rest spring and d_o the solution of the equilibrium equation $d_o = (m_o g)/k$.

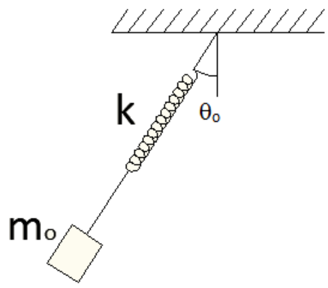

Figure 3: Physical scheme of the elastic pendulum

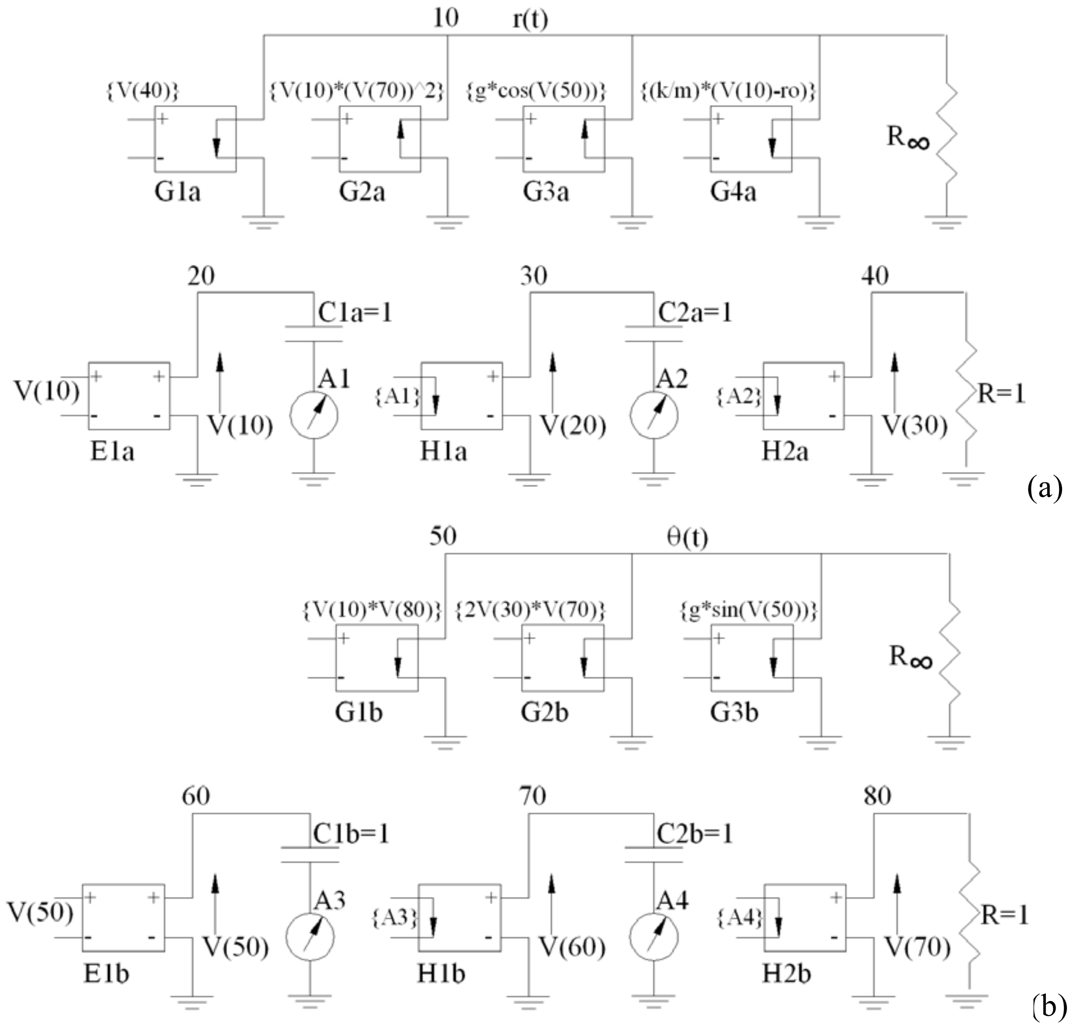

Figure 4: Network model of the elastic pendulum. (a): Equation (1), (b): equation (2)

Figure 4 shows the network model formed by two main loops (one per equation) plus the auxiliary circuits. Controlled sources G1a to G4a implement the four terms of equation

(1), the r equation, while G1b to G3b balance the three terms of equation (2), the θ equation. The control voltage of each source is shown between brackets. Auxiliary circuits E1a, H1a and H2a, together with the associated capacitors and ammeters implement the necessary electrical components required to form the successive derivative terms of the equation (1), while E1b, H1b and H2b plays this role for the equation (2).

The solution for r(t), dr/dt and d^2r/dt^2 can be read at the nodes 10 (the node of the main loop of the r circuit), 30 and 40, respectively, while the solution for θ(t), $d\theta/dt$ and $d^2\theta/dt^2$ can be read at the nodes 50 (the node of the main loop of the θ circuit), 70 and 80, respectively. Couple terms are directly written by software when specifying the control voltage of the associated source. Initial conditions for r, θ, and their derivatives dr/dt and $d\theta/dt$ are implementing by fixing the initial voltage of the capacitors C1a, C1b and C2a, C2b, respectively.

For the following values of the parameters:

$m_o = 1$, $k = 10$, $r_o = 2.2$, $g=2$, $\theta_{initial} = 0.01$, $(dr/dt)_{initial} = (d\theta/dt)_{initial} = 0$

the solution, represented in the output graphic ambient of Pspice, is shown in Figure 5. Computational time is of the order of 5 s in a portable PC.

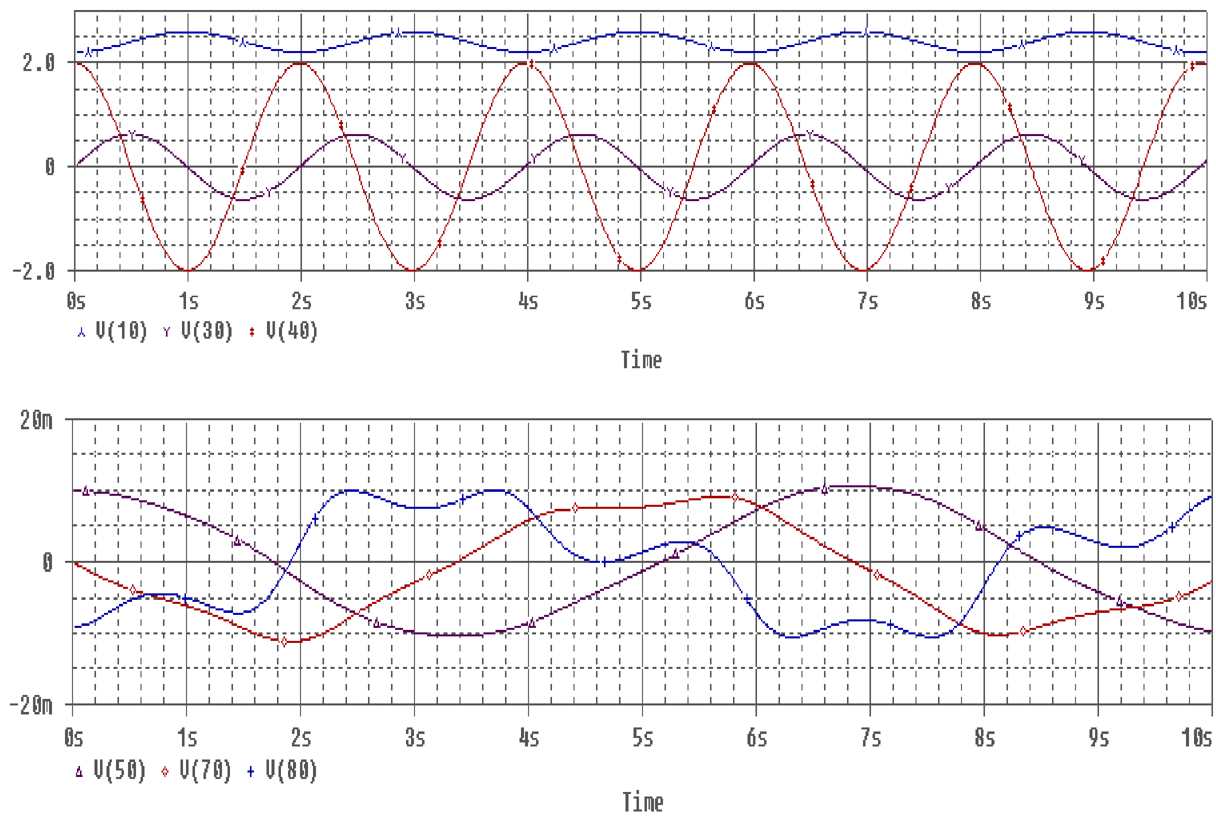

Figure 5: Solution for r(t), dr/dt and d^2r/dt^2 (up) and for θ(t), $d\theta/dt$ and $d^2\theta/dt^2$ (down)

3.2 The chaotic double pendulum

Two small masses are hanging as shows Figure 6. When one or both masses are displaced from their equilibrium location, the system oscillates in a chaotic form.

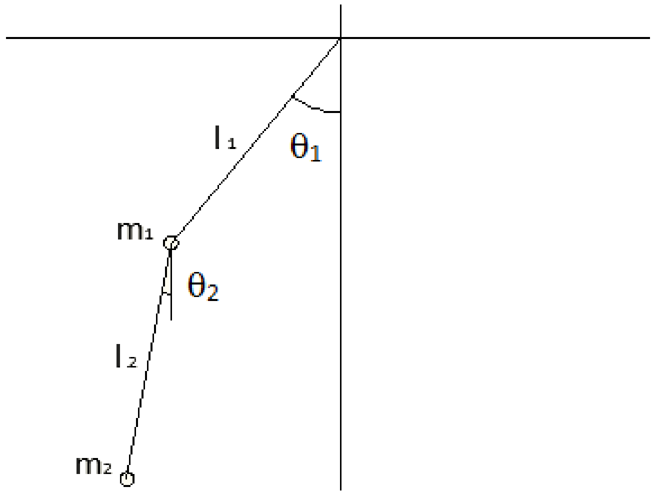

Figure 6: Physical scheme of the double pendulum

The governing equations for small displacements from the equilibrium (angles sufficiently small as to assume $\sin\theta \approx \theta$), are

$$2\left(\frac{d^2\theta_1}{dt^2}\right)+\left(\frac{d^2\theta_2}{dt^2}\right)+2\left(\frac{g}{l_o}\right)\theta_1=0 \qquad (3)$$

$$\left(\frac{d^2\theta_2}{dt^2}\right)+\left(\frac{d^2\theta_1}{dt^2}\right)+\left(\frac{g}{l_o}\right)\theta_2=0 \qquad (4)$$

Figure 7 shows the network model which can be understood following the explanation of the elastic pendulum. Auxiliary circuits are the same of the former application. Controlled sources G1a to G3a implement the terms of equation (3), the θ_1 equation, while G1b to G3b the terms of equation (4), the θ_2 equation. The solution for $\theta_1(t)$, $d\theta_1/dt$ and $d^2\theta_1/dt^2$ can be read at the nodes 10, 30 and 40, while that of $\theta_2(t)$, $d\theta_2/dt$ and $d^2\theta_2/dt^2$ at the nodes 50, 70 and 80, respectively.
For the following values of the parameters:

$m_1 = 1$, $m_2 = 1$, $l_1 = 1$, $l_2 = 1$, $g = 1$, $\theta_{1,initial} = 0.001$, $\theta_{2,initial} = 0.005$,
$(d\theta_1/dt)_{initial} = (d\theta_2/dt)_{initial} = 0$

the solution is shown in Figures 8 and 9. For a better appreciation of the chaotic movement the phase diagrams $\theta_1 = \theta_1(d\theta_1/dt)$ and $\theta_2 = \theta_2(d\theta_2/dt)$ are shown in Figures 10 and 11. Total computing time is of the order of 6 s.

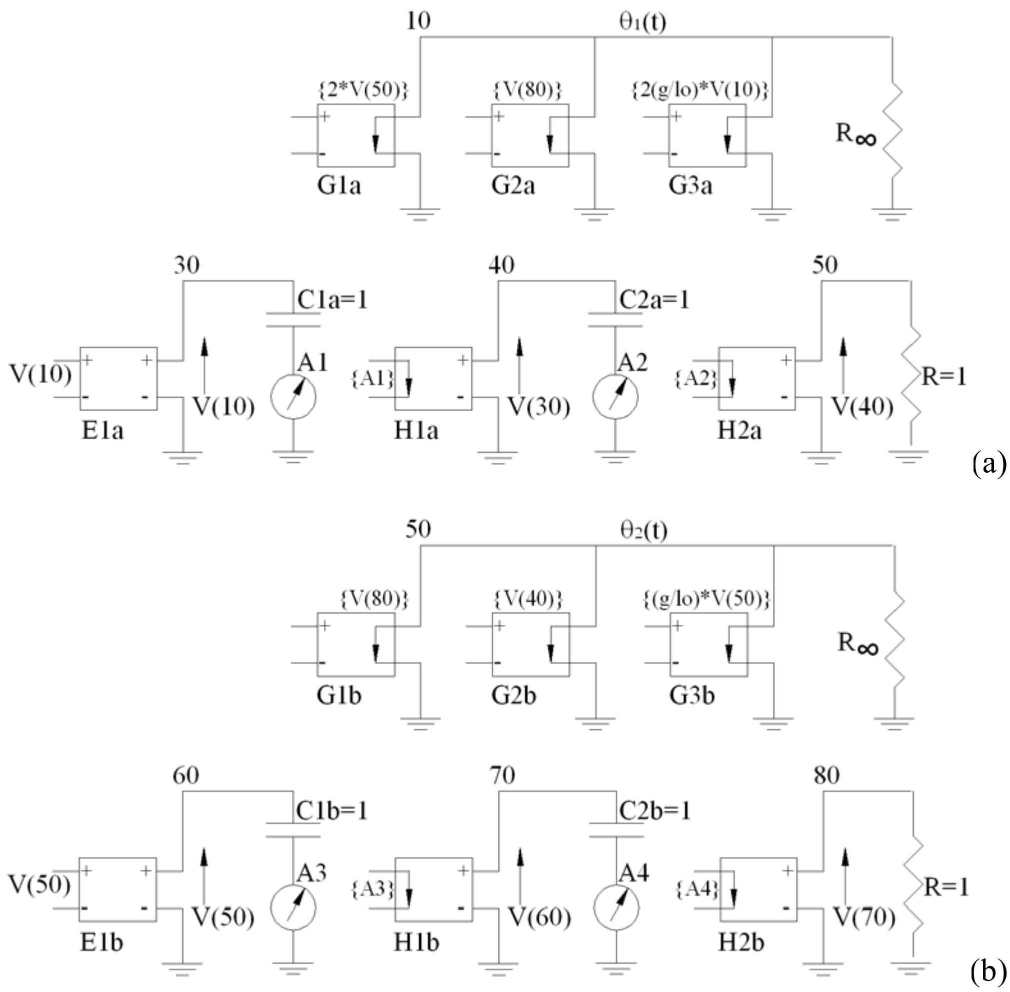

Figure 7: Network model of the double pendulum. (a): equation (3), (b): equation (4)

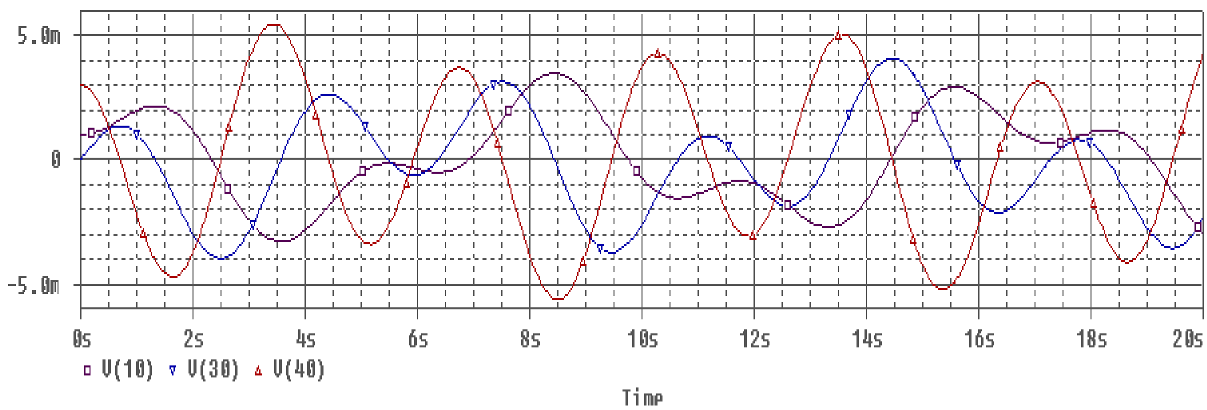

Figure 8: Solution for $\theta_1(t)$, $d\theta_1/dt$ and $d^2\theta_1/dt^2$

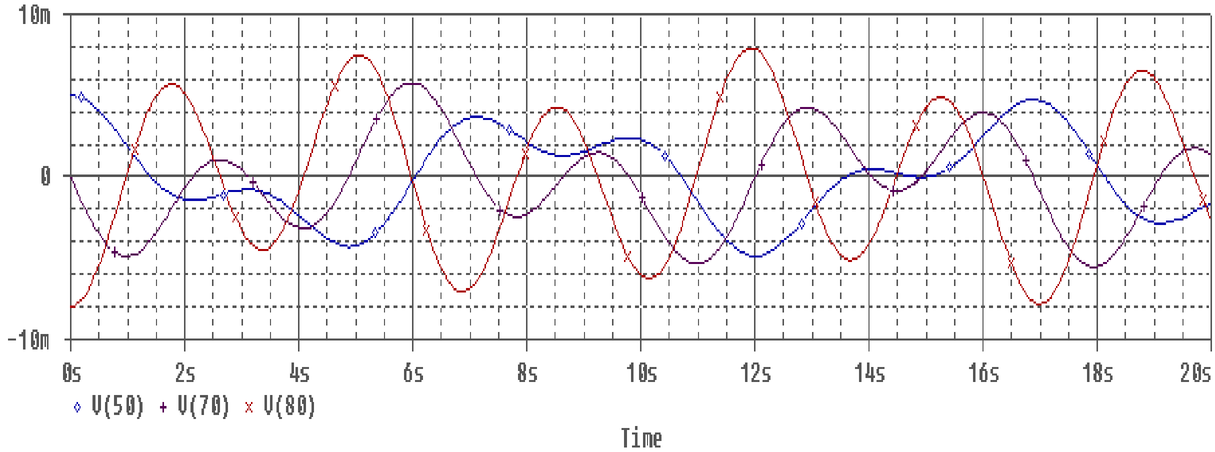

Figure 9: Solution for $\theta_2(t)$, $d\theta_2/dt$ and $d^2\theta_2/dt^2$

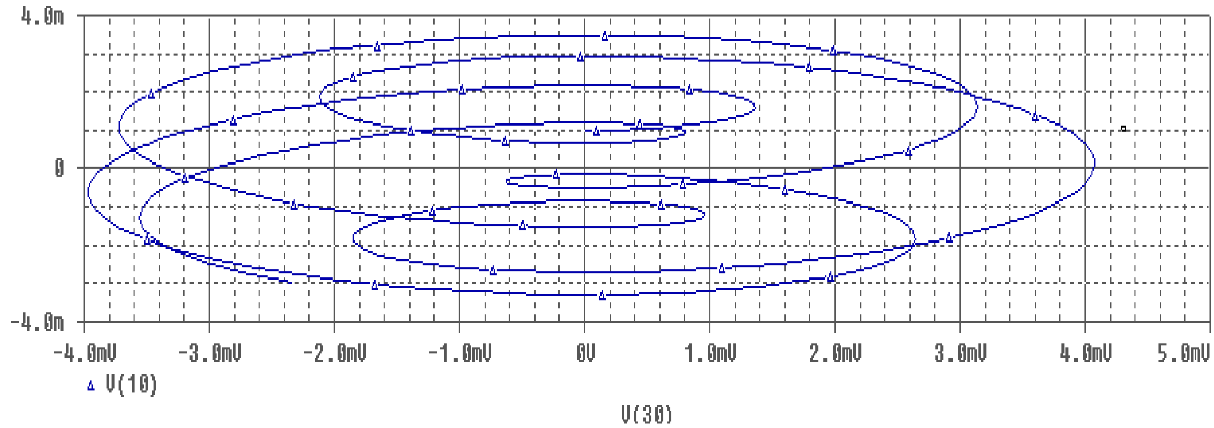

Figure 10: Phase diagrams. $\theta_1 = \theta_1(d\theta_1/dt)$

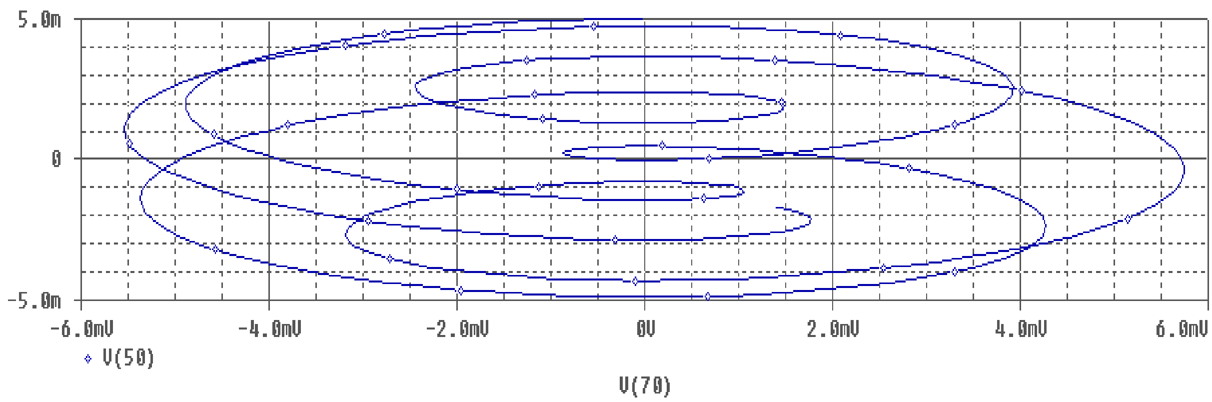

Figure 11: Phase diagrams. $\theta_2 = \theta_2(d\theta_2/dt)$

4 CONCLUSIONS

Network method has demonstrated to be an efficient tool for the numerical simulation of mechanical problems whose mathematical model is formed by coupled, non-lineal ordinary differential equations of any order and any grade. The design of the model is relatively simple since very few electrical components are required which, in turn, makes that very few programming rules are needed. Thanks to the so named 'controlled sources', a special device contained in the libraries of the circuit simulation computing codes, whose output are specified by software, any kind of nonlinearity as well as coupled terms can be easily implemented in the model. A same protocol define the design of the network; this contains as many main loops as governing equations, where the solution of the dependent variables emerges at the common nodes, and as many auxiliary loops as derivative terms contain the equations, where the derivative terms can be determined. The power mathematical algorithms used in modern codes make the computational time quite negligible and the numerical solution reliable.

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